ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 36 Principles of Digital Communications

Solution to Final exam June 16, 2024

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PROBLEM 1. (a) **No, the error probability of the new MAP rule is not higher than the equally likely case.**

Explanation :

The first case applies this to decide :

$$\hat{H}_1 = \arg\max_i P_{Y|H}(y|i)$$

The second case applies this to decide :

$$\hat{H}_2 = \arg\max_i P_{Y|H}(y|i)P_H(i)$$

The first case corresponds to the ML rule whereas the second case corresponds to the MAP rule. But the MAP rule is the one that minimizes the error probability. So it cannot give a higher error probability than the ML rule.

(b) **Yes,** $L(y) = [L_1(y), L_2(y), ..., L_{m-1}(y)]^T$ **is a sufficient statistic.**

Explanation :

One can observe that $p_i(y) = L_1(y)L_2(y)...L_i(y)p_0(y) = p_0(y)\prod_{j=1}^{i} L_j(y) = h(y)g_i(L(y))$. By Fisher-Neyman, we can directly conclude that $L(y)$ is a sufficient statistic.

(c) **Yes, the information given is sufficient to compute the error probability.**

We need the distance between each waveform. Let us call d_{ij} the distance between w_i and w_j.

$$d_{ij}^2 = \|w_i - w_j\|^2 = \langle w_i - w_j, w_i - w_j\rangle = \langle w_i, w_i\rangle - 2\langle w_i, w_j\rangle + \langle w_j, w_j\rangle = A_{ii} - 2A_{ij} + A_{jj}$$

Now that we have how to compute the pairwise distances between waveforms, we can compute the error probability using it. In fact, the exact MAP rule can be implemented by just knowing the matrix A.

(d) **Yes,** ψ **is a Nyquist pulse with parameter 2T.**

Explanation :

$$\begin{aligned}\sum_n |\psi_F\left(f - \frac{n}{2T}\right)|^2 &= \sum_k |\psi_F\left(f - \frac{2k}{2T}\right)|^2 + \sum_k |\psi_F\left(f - \frac{2k+1}{2T}\right)|^2 \\ &= \sum_k |\psi_F\left(f - \frac{k}{T}\right)|^2 + \sum_k |\psi_F\left(f - \frac{k}{T} - \frac{1}{2T}\right)|^2\end{aligned}$$

Now, we can set $f' = f - \frac{1}{2T}$ and continue :

$$\sum_n |\psi_F\left(f - \frac{n}{2T}\right)|^2 = \sum_k |\psi_F\left(f - \frac{k}{T}\right)|^2 + \sum_k |\psi_F\left(f' - \frac{k}{T}\right)|^2 = T + T = 2T$$

One can also observe that

$$\langle \psi(\cdot - nT), \psi(\cdot - mT)\rangle = \delta_{mn} \ \forall m, n \in \mathbb{Z} \implies \langle \psi(\cdot - i2T), \psi(\cdot - k2T)\rangle = \delta_{ik} \ \forall i, k \in \mathbb{Z}$$

by choosing $n = 2i$ and $m = 2k$ for some $i, k \in \mathbb{Z}$. This means that ψ is orthogonal to its $2T$-translates, which equivalently means that it is a Nyquist pulse with parameter $2T$.

Problem 2. (a) We have the following set-up :

$$\text{Under } H = i : Y = c_i + Z \text{ where } c_i = (-1)^i[A, \ldots, A] \text{ and } Z \sim \mathcal{N}\left(0, \frac{N_0}{2} I_n\right)$$

Let d be the distance between c_1 and c_0. We need it to find where the decision boundary between the two point will be placed.

$$d = \|c_0 - c_1\| = \sqrt{4A^2 \times n} = 2A\sqrt{n}$$

Now that we have the distance between the two points and knowing that the messages are equally likely, we can compute P_e as :

$$P_e = \Pr\left(Z > \frac{d}{2}\right) = Q\left(\frac{2A\sqrt{n}}{\sqrt{2N_0}}\right) = Q\left(A\sqrt{\frac{2n}{N_0}}\right)$$

(b) Let us express the error probability in terms of the new variables N_0' and A'.

$$d = \|c_0 - c_1\| = \sqrt{4A'^2 \times n} = 2A'\sqrt{n}$$

$$P_{e,new} = \Pr\left(Z > \frac{d}{2}\right) = Q\left(\frac{2A'\sqrt{n}}{\sqrt{2N_0'}}\right) = Q\left(A'\sqrt{\frac{2n}{4N_0}}\right) = Q\left(A'\sqrt{\frac{n}{2N_0}}\right)$$

To make the error probability unchanged, we need

$$A'\sqrt{\frac{n}{2N_0}} = A\sqrt{\frac{2n}{N_0}} \Rightarrow A' = 2A$$

Thus, $A' = 2A$ makes the error probability unchanged.

(c) Same reasoning as in the previous question. This time, we need :

$$A\sqrt{\frac{n'}{2N_0}} = A\sqrt{\frac{2n}{N_0}} \Rightarrow n' = 4n$$

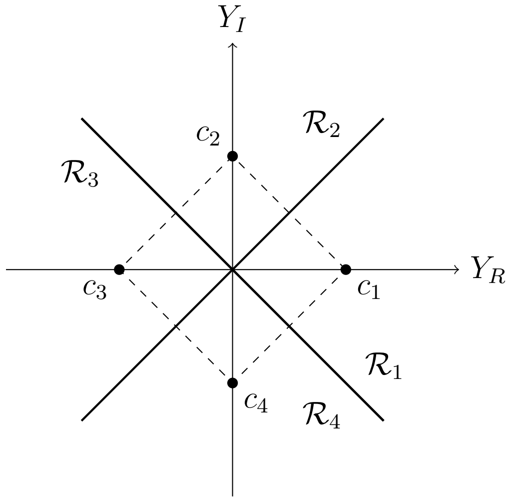
Thus, $n' = 4n$ makes the error probability unchanged

(d) In question a) we found : $P_e = Q\left(A\sqrt{\frac{2n}{N_0}}\right) \approx \exp\left(-\frac{A^2 n}{N_0}\right)$.
Now, as N_0 has changed to $N_0/4$, we have as a new error probability :

$$\tilde{P}_e = Q\left(A\sqrt{\frac{8n}{N_0}}\right) \approx \exp\left(-\frac{4A^2 n}{N_0}\right) \approx P_e^4$$

Problem 3. (a) Block 1 : Computes $Y_j = \langle r_E, \psi(t-j)\rangle$ using a matched filter with $h(t) = \psi(-t)$ sampled at $t = 0, 1, \ldots$

Block 2 : $Y_j = X_j + \langle N_E, \psi(t-j)\rangle = X_j + Z_R + \mathrm{j}Z_I$ where $Z_R = \langle N_R, \psi(t-j)\rangle$ and $Z_I = \langle N_I, \psi(t-j)\rangle$ are $\mathcal{N}\left(0, \frac{N_0}{2}\right)$ i.i.d. Let Y_R be the real part of Y_j and Y_I be the imaginary part. Then, the decision regions are given on the complex plane by the following, where $c_1 = +\sqrt{\mathcal{E}_s},\ c_2 = \mathrm{j}\sqrt{\mathcal{E}_s},\ c_3 = -\sqrt{\mathcal{E}_s},\ c_4 = -\mathrm{j}\sqrt{\mathcal{E}_s}$:



The decision $\hat{X}_j$ is c_i if Y_j lies in $\mathcal{R}_i$.

(b) For any of the 4 points, its distance to each of the two boundaries is $\sqrt{\mathcal{E}_s}\sin\left(\theta - \frac{\pi}{4}\right)$ and $\sqrt{\mathcal{E}_s}\cos\left(\theta - \frac{\pi}{4}\right)$. Let us say that the noise added is $Z = (Z_1, Z_2)$, where Z_1 is the direction parallel to one boundary and Z_2 to the other. If c_i is sent, to make the right guess, Z_1 should be greater than $-\sqrt{\mathcal{E}_s}\sin(\frac{\pi}{4} - \theta)$, otherwise it will make you outside of the right decision region, and Z_2 should be greater than $-\sqrt{\mathcal{E}_s}\cos(\frac{\pi}{4} - \theta)$ otherwise it will also make you outside of the right decision region. This is the same as :

$$\begin{aligned}
\Pr(\text{error}) &= 1 - \Pr(\text{correct}) \\
&= 1 - \Pr\left(Z_1 > -\sqrt{\mathcal{E}_s}\sin\left(\frac{\pi}{4} - \theta\right)\right) P\left(Z_2 > -\sqrt{\mathcal{E}_s}\cos\left(\frac{\pi}{4} - \theta\right)\right) \\
&= 1 - \left(1 - Q\left(\frac{\sqrt{\mathcal{E}_s}\sin\left(\frac{\pi}{4} - \theta\right)}{\sigma}\right)\right)\left(1 - Q\left(\frac{\sqrt{\mathcal{E}_s}\cos\left(\frac{\pi}{4} - \theta\right)}{\sigma}\right)\right) \\
&= Q\left(\frac{\sqrt{\mathcal{E}_s}\sin(\frac{\pi}{4} - \theta)}{\sigma}\right) + Q\left(\frac{\sqrt{\mathcal{E}_s}\cos(\frac{\pi}{4} - \theta)}{\sigma}\right) \\
&\quad - Q\left(\frac{\sqrt{\mathcal{E}_s}\sin\left(\frac{\pi}{4} - \theta\right)}{\sigma}\right) Q\left(\frac{\sqrt{\mathcal{E}_s}\cos\left(\frac{\pi}{4} - \theta\right)}{\sigma}\right)
\end{aligned}$$

(c) What would happen is that, even when there is no noise, the point is not even in its right decision region. So even with no noise, the decoder decodes incorrectly. In fact, this system (with noise) performs even worse than a random guess with error probability 3/4 (since there is a large probability that it ends up in a different region).

(d) Here, as $T = 1$, the minimum value for B is $B = \frac{1}{2}$, since otherwise, the Nyquist criterion for orthonormality would be violated.

With $\psi(0) = 1$, the corresponding pulse shape is $\psi(t) = \text{sinc}(t)$. (Any other pulse shape would lead to a larger bandwidth than 1/2.)

Problem 4. We have the following setup : Under $H = i$, $Y = c_i + Z$ where $c_i \in \{+1, -1\}^n$ and $Z \in \mathbb{R}^n$ with i.i.d components whose pdf is $f_{Z_i}(z_i) = \frac{1}{2\sigma}\exp\left(\frac{-|z_i|}{\sigma}\right)$.

(a) Applying the ML rule (because equivalent to the MAP rule, as the messages are equally likely) we have :

$$\begin{aligned}
\hat{H}_{ML} &= \arg\max_i \; f_{Y|H}(y|i) \\
&= \arg\max_i \; f_Z(y - c_i) \\
&= \arg\max_i \; \prod_{j=1}^{n} \frac{\exp\left(\frac{-|y_j - c_{i,j}|}{\sigma}\right)}{2\sigma} \\
&= \arg\min_i \; \sum_{j=1}^{n} |y_j - c_{i,j}|
\end{aligned}$$

(b) Let us rewrite the rule found in a) using the additional information given in this question.

$$\begin{aligned}
\hat{H}_{ML} &= \arg\min_i \; \sum_{j=1}^{n} |y_j - c_{i,j}| \\
&= \arg\min_i \; \sum_{j=1}^{n} |y_j| + |c_{i,j}| - 2\min(|y_j|, |c_{i,j}|)\mathbb{1}(y_j c_{i,j} > 0) \\
&= \arg\min_i \; \sum_{j=1}^{n} 1 - 2\min(|y_j|, |c_{i,j}|)\mathbb{1}(y_j c_{i,j} > 0) \\
&= \arg\max_i \; \sum_{j=1}^{n} 2\min(|y_j|, |c_{i,j}|)\mathbb{1}(y_j c_{i,j} > 0) \\
&= \arg\max_i \; \sum_{j=1}^{n} \min(|y_j|, 1)\mathbb{1}(y_j c_{i,j} > 0)
\end{aligned}$$

(c) We can compute:

$$\begin{aligned}
\Pr(\text{error}|i) &\leq \sum_{j\neq i} \sqrt{\frac{\Pr_H(j)}{\Pr_H(i)}} \int_{\mathbb{R}^n} \sqrt{f_{Y|H}(y|i) f_{Y|H}(y|j)}dy \\
&= \sum_{j\neq i} \int_{\mathbb{R}^n} \sqrt{\frac{1}{(2\sigma)^n} e^{-\frac{\|y-c_i\|_1}{\sigma}} \frac{1}{(2\sigma)^n} e^{-\frac{\|y-c_j\|_1}{\sigma}}} dy \\
&= \sum_{j\neq i} \frac{1}{(2\sigma)^n} \int_{\mathbb{R}^n} \sqrt{e^{-\frac{1}{\sigma}(\|y-c_i\|_1+\|y-c_j\|_1)}}dy \\
&= \sum_{j\neq i} \frac{1}{(2\sigma)^n} \int_{\mathbb{R}^n} \sqrt{e^{-\frac{1}{\sigma}\sum_{k=1}^n |y_k-c_{i,k}|+|y_k-c_{j,k}|}}dy \\
&= \sum_{j\neq i} \prod_{k=1}^{n} \frac{1}{(2\sigma)^n} \int_{\mathbb{R}} e^{-\frac{1}{2\sigma}|y_k-c_{i,k}|+|y_k-c_{j,k}|}dy_k
\end{aligned}$$

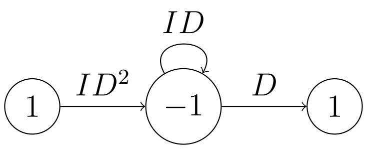
If $c_{i,k} = c_{j,k}$ then $\frac{1}{2\sigma}\int_{\mathbb{R}} e^{-\frac{1}{2\sigma}|y_k-c_{i,k}|+|y_k-c_{j,k}|}dy_k = \frac{1}{2\sigma}\int_{\mathbb{R}} e^{-\frac{1}{\sigma}|y_k-c_{i,k}|}dy_k = 1$ because it is the integral over $\mathbb{R}$ of a probability density. Therefore, the elements that are different from 1 in the product are those where $c_{i,k} \neq c_{j,k}$ and as one is necessarily +1 and the other -1, the non-zero elements in the product are of the form :

$$\frac{1}{2\sigma} \int_{\mathbb{R}} e^{-\frac{1}{2\sigma}|y_k+1|+|y_k-1|}dy_k = g(\sigma)$$

And there are $d_H(c_i, c_j)$ of them, that corresponds to the number of places where c_i and c_j differ. Consequently :

$$\Pr(\text{error}|i) \leq \sum_{j\neq i} g(\sigma)^{d_H(c_i,c_j)}$$

(d) The detour flow graph is the following :



(e) Using what we have found in question a), the branch metric that should be used by the Viterbi decoder is the following : $d(x_i, y_i) = -|y_i - x_i|$. With this metric, the Viterbi decoder should use the path with biggest value.

(f) Let us define the transfer function $T_1(I, D)$ at state (-1). We have the following system :

$$\begin{cases} ID^2 + IDT_1 = T_1 \\ DT_1 = T(I,D) \end{cases} \Rightarrow \begin{cases} T_1 = \frac{ID^2}{1-ID} \\ DT_1 = T(I,D) \end{cases} \Rightarrow DT_1 = \frac{ID^3}{1-ID} = T(I,D)$$

Let us now get the derivative of $T(I, D)$ in terms of I, at $I = 1$ and $D = z$ where $z = \sum_{i=0}^{m-1} \sum_{l \neq i} g(\sigma)^{d_H(c_i, c_l)}$ is the Bhattacharyya bound :

$$\left.\frac{\partial T(I, D)}{\partial I}\right|_{I=1, D=z} = \left.\frac{ID^3}{1 - ID}\right|_{I=1, D=z} = \frac{z^3(1-z) + z^4}{(1-z)^2} = \frac{z^3}{(1-z)^2}$$

Now we can upper bound the error probability by our result :

$$P_e \leq \frac{z^3}{(1-z)^2} \text{ where } z = \sum_{i=0}^{m-1} \sum_{l \neq i} g(\sigma)^{d_H(c_i, c_l)}$$